TamraliptaMahavidyalaya

(Dept. of Mathematics)

Subject:- Ring Theory Teacher:- M.M.

Assignment-1

1. Give an example of a finite noncommutative ring. Give an example of an infinite noncommutative ring that does not have a unity.
2. The ring $\{\overline{0},\overline{2},\overline{4},\ \overline{6},\overline{8}\}$ under addition and multiplication modulo 10 has a unity. Find it.
3. Give an example of a subset of a ring that is a subgroup under addition but not a subring.
4. Show, by example, that for fixed nonzero elements a and b in a ring, the equation $ax = b$ can have more than one solution. How does this compare with groups?
5. Find an integer n that shows that the rings $\mathbb{Z}_n$ need not have the following properties that the ring of integers has.
   (i) $a^2 = a$ implies $a = 0$ or $a = 1$.(ii) $ab = 0$ implies $a = 0$ or $b = 0$
   (ii) $ab = ac$ and $a \neq 0$ imply $b = c$.
6. Show that if n is an integer and a is an element from a ring, then $n \cdot (-a) = -(n \cdot a)$.
7. Show that a ring that is cyclic under addition is commutative.
8. Let a belong to a ring R. Let $S = \{x \in R\ |ax = 0\}$. Show that S is a subring of R.
9. Suppose that a and b belong to a commutative ring R with unity. If a is a unit of R and $b^2 = 0$, show that $a + b$ is a unit of R.
10. Give an example of ring elements a and b with the properties that $ab = 0$ but $ba \neq 0$.
11. Let n be an integer greater than 1. In a ring in which $x^n = x$ for all x, show that $ab = 0$ implies $ba = 0$.
12. Suppose that R is a ring such that $x^3 = x$ for all x in R. Prove that $6x = 0$ for all x in R.
13. Suppose that a belongs to a ring and $a^4 = a^2$. Prove that $a^{4n} = a^{2n}$ for all $n \geq 1$.
14. Find an integer $n > 1$ such that $a^n = a$ for all a in $\mathbb{Z}_6$. Do the same for $\mathbb{Z}_{10}$. Show that no such n exists for $\mathbb{Z}_m$ when m is divisible by the square of some prime.
15. Let m and n be positive integers and let k be the least common multiple of m and n. Show that $mZ \cap nZ = kZ$.
16. Explain why every subgroup of $\mathbb{Z}_n$ under addition is also a subring of $\mathbb{Z}_n$.
17. Is $\mathbb{Z}_6$ a subring of $\mathbb{Z}_{12}$?
18. Suppose that R is a ring with unity 1 and a is an element of R such that $a^2 = 1$. Let $S = \{ara\ |r \in R\}$. Prove that S is a subring of R. Does S contain 1?
19. Show that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2}|a,b \in \mathbb{Q}\}$ is a subring of R.
20. Prove that the only subring of $\mathbb{Z}_n$ is itself.
21. Which of the sets described in Exercises 8.16 to 8.20 are subrings of $\mathbb{C}$? Give reasons.
    $$(i)\{0 + ib|b \in \mathbb{R}\}.(ii)\{a + ib|a,b \in \mathbb{Q}\}(iii)\left\{a + b\sqrt{-7}\ \middle|a,b \in \mathbb{Z}\right\}.$$
    $$(iii)\{z \in \mathbb{C}||z| \leq 1\}.(iv)\{a + ib|a,b \in \mathbb{Z}\}.$$
22. Let $M2(\mathbb{Z})$ be the ring of all 2×2 matrices over the integers and let $R = \left\{\begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix} : a,b \in \mathbb{Z}\right\}$. Prove or disprove that R is a subring.

23. Let $M2(\mathbb{Z})$ be the ring of all 2×2 matrices over the integers and let $R = \left\{\begin{pmatrix} a & a-b \\ a-b & b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$. Prove or disprove that R is a subring.
24. Let $M2(\mathbb{Z})$ be the ring of all 2×2 matrices over the integers and let $R = \left\{\begin{pmatrix} a & a \\ b & b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$. Prove or disprove that R is a subring.
25. Show that $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of $\mathbb{Z}$.
26. Determine the smallest subring of $\mathbb{Q}$ that contains $\frac{1}{2}$. (That is, find the subring S with the property that S contains $\frac{1}{2}$ and, if T is any subring containing $\frac{1}{2}$, then T contains S.)
27. Determine the smallest subring of $\mathbb{Q}$ that contains $\frac{2}{3}$.
28. Let R be a ring. Prove that $a^2 - b^2 = (a+b)(a-b)$ for all a, b in R if and only if R is commutative.
29. Give an example of a Boolean ring with four elements. Give an example of an infinite Boolean ring.
30. Prove that the intersection of any set of ideals of a ring is an ideal.