Riemann Integration

1. Let $f:[a,b]$ ® ¡ be bounded over $[a\ ,\ b]$. Define upper sum $U(P\ ,\ f)$, lower sum $L(P\ ,\ f)$ for a partition P over $[a\ ,\ b]$. Deduce that $m(b-a)\ \le\ L(P\ ,\ f)\ \le\ U(P\ ,\ f)\ \le\ M(b-a)$. Where $m=\operatorname*{Inf}_{x\in[a,b]} f(x)\ M=\operatorname*{Sup}_{x\in[a,b]} f(x)$. Define the upper integral & the lower integral of f over$[a\ ,\ b]$. When f is said to be R-integrable over $[a\ ,\ b]$.
2. Define upper sum $U(P\ ,\ f)$, lower sum $L(P\ ,\ f)$ of a bounded function f defined on $[a\ ,\ b]$ corresponding to a partition P of $[a\ ,\ b]$. If Q be a refinement of a partition P of $[a\ ,\ b]$ then show that Deduce that $L(P,\ f)\le\ L(Q\ ,\ f)\ \le\ U(Q\ ,\ f)\ \le\ U(P\ ,\ f)$. Hence deduce that the lower Sum can exceed any upper Sum.
3. If f(x) be Riemann integrable in $[a\ ,\ b]$ then prove that $\int_{\underline{a}}^{b} f\,dx \le \int_{a}^{\overline{b}} f\,dx$
4. If, where $[x]$ denotes the greatest integer noumber greater than the real number x, prove that $f(x)$ is integrable on [0, 3] & $\int_0^3 f\,dx = 3$.
5. Let $f:[a,b]$ ® ¡ be bounded on the closed and bounded interval $[a\ ,\ b]$. Prove that to every pre-assigned number $\varepsilon\ >\ 0$ there corresponds a positive δ such that $U(P\ ,\ f)\ <\ \int_a^{\bar{b}} f\,dx\ +\ \varepsilon$ for every partition P over $[a\ ,\ b]$ satisfying $\|\ P\ \|\ <\ \delta$, (symbals used have their usual meaning).
6. **L**et $f:[a,b]$ ® ¡ be bounded over $[a\ ,\ b]$. Prove that f is Riemann integrable on $[a\ ,\ b]$ iff for each $\varepsilon\ >\ 0$, there corresponds a partition P on $[a\ ,\ b]$ such that $U(P\ ,\ f)\ -\ L(P\ ,\ f)\ <\ \varepsilon$
7. Let the function $f:[0,1]$ ® ¡ be defined by $f(x)\ =\ \sqrt{1-x^2}$, when x is rational ,= $1\ -\ x$, when x is irrational . Is f Riemann-integrable over [0,1]? Justifu?
8. Prove that a bounded function f on $[a\ ,\ b]$ is Riemann integrable iff given any $\varepsilon\ >\ 0$, there is a $\delta>0$ such that $U(P\ ,\ f)\ -\ L(P\ ,\ f)\ <\ \varepsilon$ for all partition P of $[a\ ,\ b]$ with $\|P\|<\delta$
9. Show that the function f define by $f(x)\ =\ 0$ when x is rational ,
   $=\ 1$ when x is irrational .
   Calculate the upper & lower integral f(x) on [0,1]. Is f integrable on [0,1]?
10. If $f(x)\ =\ 3$ when x is rational in [1,5], $=\ 2$ when x is irrational in [1,5]. Show that f(x) is not integrable on [1,5].
11. Prove that every continuous function defined on the closed interval $[a\ ,\ b]$ is Riemann integrable over $[a\ ,\ b]$.

12. If a function $f:[a,b]\to\mathbb{R}$ be continuous $[a, b]$, prove that f is Riemann integrable on $[a, b]$.If a function f is Riemann integrable on $[a, b]$., is f necessarily continuous over [a, b]? Justify your answer

13. Construct, with proper justification, a bounded function, while is not integrable in the Riemann sense . [

14. Prove that if a function f is monotonic on $[a, b]$ then it is Riemann integrable on $[a, b]$.

15. Let $f:[a,b]\to\mathbb{R}$ be bounded on $[a, b]$ and let f be continuous on $[a, b]$ except for a finite number number of points in (a, b).Show that f is Riemann integrable on $[a, b]$.

16. Prove or disprove: A bounded real valued function defined on the interval $[-100, 100]$ which s discontinuous only at the integers points, is Riemann integrable on the imterval.

17. Let $f:[a,b]\to\mathbb{R}$ be bounded function such that the derived set of the set of all points of discontinuity if f is a finite set contained in (a, b). Show that f is Riemann integrable on $[a, b]$.

18. Give examples of a monotonic and a non monotonic fuction on $[0, 1]$ with infinitely many points of discontinuity such that the function are bounded and Riemann integrable on $[0, 1]$.

19. Let f(x) be defined on [0, 1] as, $f(x)=\frac{1}{2^n}$, where $\frac{1}{2^{n+1}}<x\le\frac{1}{2^n}$, n=0,1,2,3,...... Show that f(x) has
=0, when x=0.
infinite nimber of points of discontinuity on [0,1] but it is Riemann Integrable on [0,1].

20. If $f:[a,b]\to\mathbb{R}$ is bounded on the closed and bounded interval $[a, b]$ and be R-integrable on $[c, d]$ for every closed interval $[c, d]$ such that $a<c<d<b$ then Prove that f is integrable in $[a, b]$.

21. Let $f, g:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a, b]$. We define $(f+g)(x) = f(x) + g(x)$ $\forall x\in[a, b]$. Show that $f+g$ is Riemann integrable in $[a, b]$

22. Let f and g are both Riemann integrable functions on $[a, b]$. Prove that the function fg defined by then $(fg)(x) = f(x)\cdot g(x)$ for every $x\in[a, b]$, is Riemann integrable on $[a, b]$.

23. Let $f, g \in \mathbb{R}[a,b]$ $[a, b]$ then prove that $\left\{\int_a^b fg\,dx\right\}^2 \le \left(\int_a^b f^2dx\right)\left(\int_a^b g^2dx\right)$.

24. If $f:[a,b]\to\mathbb{R}$ is R-integrable on $[a, b]$, prove that $|f|$ is Riemann integrable on $[a, b]$. Give an example to show that the converse is not true.

25. Let f be Riemann integrable on $[a, b]$. If g be a bounded function defined on $[a, b]$ such that $f=g$ except for at most a finite number of points, then g is also Riemann integrable on $[a, b]$.

26. Without assuming the theorem that product of two R-integrable functions defined over an interval I is R-integrable on I, prove that if f be R-inegrable functionon the closed and bounded interval is $[a, b]$ then f^2 is also R-integrable on $[a, b]$.

27. Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a, b]$ and infimum of f is a positive real number. Prove that the function $\frac{1}{f}$ is Riemann integrable on $[a, b]$.

28.Let $f:[a,b]$ ® ¡ be integrable on $[a,b]$ and $f(x)\geq 0\ \forall\ x\in[a,b]$. If $\exists$ a point $c\in[a,b]$ such that f is continuous at the point $x=c$ and $f(c)>0$, then prove that $\int_a^b f(x)dx>0$.

29.If f is non-negetive continuous function on $[a,b]$ and $\int_a^b f(x)dx=0$ then prove that $f(x)=0\ \forall\ x\in[a,b]$.

30.If f,g be both continuous on $[a,b]$ and $\int_a^b f(x)dx=\int_a^b g(x)dx$ then prove that $\exists\ c\in[a,b]$ such that $f(c)=g(c)$.

31.If f be a continuous function on the closed and bounded imterval $[a,b]$ and if $\int_a^b f(x)g(x)dx=0$ for every continuous function g on $[a,b]$ then show that $f(x)=0$ $\forall\ x\in[a,b]$.

32.Let $f:[a,b]$ ® ¡ be integrable on $[a,b]$ then prove that $\int_a^b |f(x)|dx>\left|\int_a^b f(x)dx\right|$. Give an example where equality holds.

33.If $f:[a,b]$ ® ¡ is integrable on $[a,b]$ and F be defined by $F(x)=\int_a^x f(x)dx\ \ a\leq x\leq b$, show that F is continuous on $[a,b]$. If, in addition, f is continuous on $[a,b]$, show that $F^{(1)}(x)=f(x)$ on $[a,b]$.

34.Let $f:[a,b]$ ® ¡ be integrable on $[a,b]$. Then the function ϕ defined by $\phi(x)=\int_a^x f(x)dx$ $\forall\ x\in[a,b]$ is continuous on $[a,b]$. Furthermore if f is continuous on $[a,b]$ then at every point of $[a,b]$ ϕ is differentiable and $\phi^{(1)}(x)=f(x)\ \forall\ x\in[a,b]$.

35.Show that the function $f:[-1,1]$ ® ¡ be defined by

$$\begin{aligned} f(x) &= 2x\sin\frac{1}{x^2}-\frac{1}{x}\cos\frac{1}{x^2}, \quad x\neq 0 \\ &= 0, \quad x=0 \end{aligned}$$

possess primitive but f is not Riemann Integrable on [-1,1].

36.Evaluate $\int_0^1 (2x\sin\frac{1}{x}-\cos\frac{1}{x})dx$

37.Distinguish between an integral and a primitive of a function. Show that the function f defined on [-1, 1], by $f(x) = 2x\sin\frac{1}{x^2} - \frac{1}{x}\cos\frac{1}{x^2}, \quad x \neq 0$
$= 0, \quad x = 0$ possess a primitive but f is not integrable.

38.If f be bounded and integrable on $[a, b]$ and $\exists$ a function F such that $F^{/}(x) = f(x)$ $\forall x \in [a, b]$, prove that $\int_a^b f(x)dx = F(b) - F(a)$. Give an example of a which is Riemann integrable without having a primitive.

39.State and prove the fundamental theorem of integral calculus. Deduce that
$\int_a^b f(x)\, dx = (b-a)f[a+\theta(b-a)],\ 0<\theta<1$ under suitable condition on f(x) stated by you

40.Show that evaluation of $\int_0^2 f(x)dx$ where $f(x)=[x]$, $x \in [0,2]$ cant be done by the fundamental theorem of Integral Calculus.

41.(i) f and f are bounded and integrable in $[a, b]$ and ϕ is continuous at least of one point in $[a, b]$ and is of the same sign of it , then show that there exists a real number $\mu \in [m, M]$ so that $\int_a^b f(x)\,\phi(x)\,dx = \mu\int_a^b \phi(x)dx$ where $m \le \mu \le M$, $m = \inf\{ f(x) : x \in [a, b]\}$ and $M = \sup\{ f(x) : x \in [a, b]\}$.

42.Let ϕ be bounded and integrable and maintains the same sign on $[a, b]$ and f is continuous on $[a, b]$, prove that $\int_a^b f(x)\,\phi(x)\,dx = f\{a+\theta(b-a)\}\int_a^b \phi(x)dx$ for some $\theta \in [0,1]$

43. Let $\phi : [a, b] \to$ R be integrable on $[a, b]$ and ϕ has the same sign for all $x \in [a, b]$. If $f : [a, b]$ ® ¡ is continuous on $[a, b]$, then there exists at lest one point $\xi \in [a, b]$ such that
$\int_a^b f(x)\,\phi(x)\,dx = f(\xi)\int_a^b \phi(x)dx$

44.State and prove 1^{st} mean value theorem of Integral calculus.

45.Let a function f is defined on $[a, b]$ by $f(x)=e^x$. Find $\int_{\bar{a}}^b f\,dx$ & $\int_a^{\bar{b}} f\,dx$. Also show that f is Riemann Integrable on $[a, b]$.

46. A function f is defined on $[0,1]$ by
$f(x) = x$, if x be rational
$= x^2$, if x be irrational.

Find $\underline{\int_0^1} f(x)\,dx$ and $\overline{\int_0^1} f(x)\,dx$. Deduce that f is not integrable on $[0,1]$.

47. A function f is defined on $[0,1]$ by $f(0)=0$ and

$$f(x)=0, \text{ if } x \text{ be irrational}$$
$$=\frac{1}{q}, \text{ if } x=\frac{p}{q} \text{ where } p,q \text{ are positive integers prime to each other.}$$

Show that f is integrable on $[0,1]$ and $\int_0^1 f(x)\,dx=0$.

48. Show that there is a value $\xi \in [a,\pi]$ $\int_0^{\pi} e^{-x}\cos x\,dx = \sin\xi$

49. Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$

50. Show that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^5}} < \frac{\pi}{6}$

51. Using first mean value theorem of Integral Calculus, Show that

$$\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} \le \frac{\pi}{6}\frac{1}{\sqrt{1-\frac{k^2}{4}}}, \left(k^2<1\right).$$

52. Show that for $\left(k^2<1\right)$ $\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} \le \frac{\pi}{6}\frac{1}{\sqrt{1-\frac{k^2}{4}}}$.

53. Show that $\frac{\sin x}{x}$ is R-integrable in $[\frac{\pi}{4},\frac{\pi}{3}]$. Hence show that $\frac{\sqrt{3}}{8} \le \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} \le \frac{\sqrt{2}}{6}$

54. With proper justification show that $\frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}$.

55. State the second Mean Value theorem of integral calculus in Weirstrass form. Evamine the valididity of this theorem for the integral $\int_{-\pi}^{\pi} x^2\cos x\,dx$

56. Show that $\frac{\pi^3}{24\sqrt{2}} \le \int_0^{\pi/2} \frac{x^2}{\sin x+\cos x} \le \frac{\pi^3}{24}$

57. State the second Mean Value theorem of integral calculus in Bonnets form. Using it show that

$$\left|\int_a^b \frac{\sin x}{x}\,dx\right| \le \frac{2}{a}, \quad 0<a<b$$

58. State the second Mean Value theorem of integral calculus in Weirstrass form and verify it for the function $\frac{x}{\sin x}$ in $[\pi, 2\pi]$.

59. Prove that $\frac{1}{4} < \int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-x^{2n}}} < \frac{1}{\sqrt{15}}$ when $n > 1$

60. State Weierstrass form of integral calculus. Show that the theorem,s is applicable to $\int_a^b \frac{\sin x}{x}\,dx$ where, $0<a<b<8$. also prove that $\left|\int_a^b \frac{\sin x}{x}\,dx\right| \le \frac{4}{a}$,

61. Using second Mean Value theorem (Weirstrass form) of integral calculus, show that

$$\left|\int_a^b \frac{\cos x}{1+x}\,dx\right| \le \frac{4}{1+a}, \quad \text{where } b>a>0$$

62. Show that second mean value theorem does not hold in $[\pi, 2\pi]$ for the function $f(x)=\cos x$ & $g(x)=x^2$

63. If α and β are positive acute angles prove that

$$\beta < \int_0^{\beta} \frac{dx}{\sqrt{1-\sin^2\alpha\,\sin^2 x}} < \frac{\beta}{\sqrt{1-\sin^2\alpha\,\sin^2\phi}}$$