RIGID Dynamics

Moment of Inertia

1. Let A,B,C,D,E,F be the moments and products of inertia with respect to a given system of three rectangular axes passing through O, then show that the moment and products of inertia of the material system about any line through O whose direction cosines are l, m, n is $Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm$
2. What is mean by principal axis of a given material system of a point? Find whether given straight line is at any point of its length a principal axis of the material system and in case the line is the principal axis find the other two principal axis. (2+6)
3. Explain what is mean by equimomental bodies .Show that a uniform plane triangular lamina is equimomental with a system of three particles placed at middle points of the sides is equal to 1/3 the mass of the triangle
4. If A, B, C, D, E, F be the moments and products of inertia of a rigid body about rectangular axes OX, OY, OZ and if l_1, m_1, n_1, l_2, m_2, n_2 be the direction cosines of two lines OL, OK at right angles to each other, then find the moments of inertia of the body about OL and product of inertia about OL and OK.
5. If A and B be the two moments of inertia of plane lamina about rectangular axes of OX and OY in the plane of the lamina and F be the product of inertia with regard to these axes, find the moments of inertia A and B and the product of inertia A of the lamina about two perpendicular lines OQ, OR in the XY-plane where OQ makes an angle with OX show that i. $A + B = A' + B'$, $A'B' - F'^2 = AB - F^2$.
   ii. The principal moments at O are equal to $\frac{1}{2}\left\{A + B \pm \sqrt{(A-B)^2 + 4F^2}\right\}$
6. find the moment of inertia of a rectangular lamina of sides 2a,2b about an axis through centre perpendicular to the lamina. (3)
7. Find the moment of inertia of a circular ring about a diameter.
8. Find moment of inertia of a rectangular parallelepiped about any edge.
9. Find the moment of Inertia of a hollow sphere about a diameter (3)
10. Find the moment of inertia of a solid sphere about a diameter
11. A solid body of density ρ, is in the shape of the solid formed by the revolution of the cardioid $r = a(1+\cos\theta)$ about the initial line; Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\frac{352}{105}\pi\rho a^5$
12. Show that the moment of inertia of a rigid solid cone, whose height is h and radius of whose base is a, is $\frac{3Ma^2}{20} g \frac{6h^2 + a^2}{h^2 + a^2}$ about a slant side and $\frac{3M}{80} g(h^2 + 4a^2)$ about a line through the center of gravity of the cone perpendicular to its axis.
13. Show that the moment of inertia of an ellipse of mass M and semi axes a and b about a tangent is $\frac{5M}{4} p^2$. ,where p is the perpendicular from the center on the tangent.

14. If the vertical angle of a cone 90 , then show that the point , at which a generator is a principle axis , divides the generator in the ratio 3:7
15. Show that the moment of inertia of a homogeneous triangular lamina ABC about a line through A and perpendicular to the plane of the triangle is $\frac{M}{12}(3b^2+3c^2-a^2)$ where a, b, c are the lengths of sides BC ,CA and AB and M is mass of the triangle lamina .
16. Show that the momental ellipsoid at a point on the rim of a hemisphere is $2x^2+7(y^2+z^2)-\frac{15}{4}xz^2=$ constant.
17. Show that the equation of the momental ellipsoid at the corner of a cube of side 2a, referred to its principal axes is $2x^2+11(y^2+z^2)=cons\tan t$
18. Show that the momental ellipsoid at a point on the edge of the circular base of a thin hemisphere shell is $2x^2+5(y^2+z^2)-3xz=cons\tan t$
19. Show that the centre of the quadrant of an ellipse the principal axes in its plane are inclined at an angle $\frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\frac{ab}{a^2-b^2}\right)$ to the axes
20. If the Principal axis of a uniform semi-circular lamina at an extremity of its bounding diameter makes an angle θ with the diameter, then prove that $8\cot 2\theta=3\pi$.
21. If ABC be a triangular area and AD is perpendicular to BC, E be the middle point of BC and O the middle point of DE. Show that BC is a principal axis of the triangle at O.
22. Prove that a uniform triangular lamina of mass M is equimomental with three particles, each of mass M/12 placed at the angular point and particle of mass 3M/4 placed at the mass of the triangle.
23. Show that a uniform plane triangle lamina is equimomental with a system of three particles placed at the middle point of the sides , each equal to the mass of the triangle .

D' Alembert's Principle

24. State D'Alembert's principle & deduce general equations of motion of a rigid body .Prove that the motion of a body about its centre of inertia is the same as it would be of the centre of inertia were fixed & the same forces acted on the body.
25. State D'Alembert's principle. Deduce the equations of motion of the centre of inertia of a rigid body & the equation of motion relative to the centre of inertia.
26. A plank of mass M is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man, of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time $\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$ where a is the length of the plank and g is the acceleration due to gravity.

27. A thin rod of length 2a revolves with uniform angular velocity ω about a vertical axis through a small joint at one extremity of the rod so that it describes a cone of semi-vertical angle α. Show that $\omega^2 = \frac{3g}{4a\cos\alpha}$.
28. A non homogeneous rod AB of length $2l$ whose density at any point is directly proportional to the distance of the point from A, is rotating with a uniform angular velocity ω about a vertical axis through A. If the rod is inclined at an angle α to the vertical, show that the value of a is either 0 or $\cos^{-1}\left(\frac{2g}{3l\omega^2}\right)$
29. A thin heavy disc can turn freely about an axis in its own plane, and this axis revolves horizontally with a uniform angular velocity ω about a fixed point on itself. Show that the inclination θ of the plane of the disc to the vertical is $\cos^{-1}\left(\frac{gh}{k^2\omega^2}\right)$, where h is the distance of the centre of inertia of the disc from the axis and k is the radius of gyration of the disc about the axis. If , $\omega^2 < \frac{gh}{k^2}$ then the plane of the disc is vertical.
30. A rod of length $2a$, is suspended by a string of length l, attached to one end. If the string and rod revolve about the vertical with uniform angular velocity, and their inclinations to be vertical be θ and φ respectively, then show that $\frac{3l}{a} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$
31. A rough uniform board of mass m & length 2a,rests on a smooth plane,& a boy of mass M , walks on it at from one end to the other. Show that the distance through which the board moves in this time is $\frac{2Ma}{m+M}$
32. A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate. What will be the motion of the centre of a board?

Motion of a rigid body about fixed axis

33. Show that the rate of changes of the angular momentum of a rigid body about the axis of rotation is equal to the sum of moments about the same axis of all forces acting on the body.
34. A rigid body is rotating about a fixed axis. Find the expression for (i) Kinetic energy of the rigid body (ii) the angular momentum about the axis of rotation of the rigid body.
35. Define a compound pendulum, centre of suspension and Centre of oscillation. Show that the centre of suspension and Centre of oscillation are convertible. (*1+1+1+4*)
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37. Define simple equivalent pendulum. Find its length.
38. A compound pendulum of mass m oscillates about a fixed horizontal axis has its centre of oscillation at C. Find the period of oscillation of the compound pendulum. Show further that the period is unaltered even if a weight is rigidly attached to the body of the pendulum at C.

39. A uniform elliptic can rotate about a horizontal axis passing through a focus and perpendicular to its plane. If the centre of oscillation be the other focus, prove that the eccentricity is $\sqrt{2/5}$.
40. A uniform rod AB is freely movable on a rough inclined plane, inclination to the horizon is i and whose coefficient of friction is μ, about a smooth pin fixed through the end A; the bar is held in the horizontal position in the plane and allowed to fall from this position. If θ be the angle through which it falls from rest , Show that $\frac{\sin\theta}{\theta} = \mu \cot i$
41. Find the length of the simple equivalent pendulum of the cone ; axis a diameter of the base.
42. A uniform elliptic lamina can rotate about a horizontal axis passes through a foci and perpendicular to its plane . Find the centre of oscillation if the eccentricity is $\sqrt{2/5}$.
43. Show that the length of the simple equivalent pendulum of a cube of side 2a oscillating about a horizontal edge as axis is $\frac{4\sqrt{2}}{3}a$.
44. A solid homogeneous cone of height h and vertical angle 2α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is $\frac{h}{5}(4+\tan^2\alpha)$
45. An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation, Prove that the eccentricity of the ellipse is $1/2$

Motion in 2-dimension

46. Prove that the kinetic energy of a rigid body of mass M moving in two dimensions is equal to the sum of the kinetic energy of a particle of mass M placed at the centre of inertia and moving with it and the kinetic energy of the body relative to the centre of inertia.
47. Find the expression of moment of momentum about the origin of a rigid body moving in two dimension.
48. An imperfectly rough sphere moves from rest down a plane inclined at an angle α to the horizontal. Discuss the motion.
49. A sphere of radius a, is projected up an inclined plane with velocity V and angular velocity ω in the sense which would cause it to roll up, if $V > aw$ and the coefficient of friction μ exceeds $\frac{2}{7}\tan\alpha$. Show that the sphere will cease to ascend after time $\frac{5V+2a\omega}{5g\sin\alpha}$ where α is the inclination of the plane to the horizontal.
50. A uniform solid sphere rolls down an inclined plane, rough enough to prevent sliding. Discuss the motion if $\mu > \frac{2}{7}\tan\alpha$ where μ is the coefficient of friction and α is the inclination of the plane to the horizon. Also explain how the principle of conservation of energy has been preserved.
51. A homogeneous sphere of radius a is rotating with an angular velocity ω about a horizontal diameter. It is then gently placed on a table whose co-efficient of friction is μ. Show that there will

be slipping at the point of contact for time $2\omega a/7\mu g$ and then the sphere will roll with angular velocity $2\omega/7$.

52. A uniform rod is held one end in contact with a rough horizontal table at an inclination α to the horizon and then relesed. If μ be the coefficient of friction, show that it will commence to slide if $\mu < \dfrac{3\sin\alpha\cos\alpha}{1+3\sin 2\alpha}$.
53. A rough uniform rod, of length 2a, is placed on a rough table at right angles to its edge. If the centre of gravity of the rod be initially at a distance b beyond the edge, then show that the rod will begin to slide when it has turned through an angle $\tan\theta = \dfrac{\mu a^2}{a^2+9b^2}$, where μ is the coefficient of friction.
54. A cylinder of mass M and radius R is held at rest in a plane inclined at an angle θ to the horizontal. When the cylinder is released it moves down the slope. Discuss the case of pure rolling motion of the cylinder. Show that when the cylinder rolls down a length l of the slope, the velocity of its centre of mass is $\frac{4}{3} gl\sin\theta$.
55. A solid circular disc of radius a is rolling up a rough plane (along the line of greatest slope) inclined at an angle α to the horizontal. If V be the velocity of the centre of the disc at an instant, show that the disc ascends a further distance $\dfrac{3V^2}{4g\sin\alpha}$ along the plane, before coming to rest.
56. A circular homogeneous plate is projected up a rough inclined plane with velocity V with no rotation, the plane of the plate being in the plane of greatest slope. Show that the plate stops rolling after a time $\dfrac{V}{\{g(3\mu\cos\alpha+\sin\alpha)\}}$ where μ is the co-efficient of friction and α is the inclination of the plane with the horizon.
57. State the principle of energy for a system of moving particle. A uniform rod of length 2a is placed with one end in contact with a smooth horizontal table and is then allowed to fall; If α be its inclination to the vertical, show that its angular velocity when it inclined at an angle θ is $\left\{\dfrac{6g}{\sigma}\cdot\dfrac{\cos\alpha-\cos\theta}{1+3\sin^2\theta}\right\}^{\frac{1}{2}}$. Find also the reaction of the table. .

Impulsive Forces

58. Define "Impulse of a force". Find the equations of motion of a rigid body moving under impulsive forces.
59. Write down the two dimensional equations of motion of a rigid body acted on by finite forces and deduce the equations of motion when the body is acted on by impulsive forces.
60. Three unequal uniform rods AB, BC, CD are freely jointed and placed in a straight line on a smooth table. The rod AB is struck at its ends by a blow which is perpendicular to its length. Find the resulting motion and show that the velocity of the centre of AB is 19 times that of CD and its angular velocity 11 times that of CD.

61. Two uniform rods AB and AC of masses m and m' respectively are freely jointed at A and laid on a smooth horizontal table is such a way that $\angle$BAC is right angle. The rod AB is suddenly struck by a blow P at B in a direction perpendicular to AB. Show that the initial velocity of a A is $\frac{2P}{4m'+m}$.
62. AB, BC are two equal similar rods freely hinged at B and lie in a straight line on a smooth table. The end A is struck by a blow perpendicular to AB. Show that the resulting velocity of A is 7/2 times velocity of B.
63. Two equal uniform rods AB and AC are freely jointed at A and they are placed on a table so as to be at right angles. The rod AC is struck by a blow at C in a direction perpendicular to AC. Show that the resulting velocities of the middle points of AB and AC are in the ratio 2:7.
64. Three equal uniform rods AB, BC and CD are hinged freely at their ends, B and C, so as to form three sides of a square and are laid on a smooth table. The end A is struck by a horizontal blow P at right angles to AB. Show that the initial velocity of A is 19 times that of D and that the impulsive actions at B and C are 5P/12 and P/12 respectively.

Conservation of Momentum

65. State and prove the principle of conservation of angular momentum.(For under impulsive forces and under finite forces).
66. A uniform circular plate is turning in its own plane about a point A on its circumference with an uniform angular velocity ω. Suddenly A is released and another point B of the circumference is fixed. Show that the angular velocity about B is $\frac{\omega}{3}(1+2\cos\alpha)$ where α is the angle that AB subtends at the centre.
67. An elliptic lamina is rotating in its own plane about one of its foci with angular velocity ω. This focus is set free and the other focus is fixed at the same instant. Show that the resulting angular velocity is ω/3, if the eccentricity of the ellipse by √2/3.
68. A lamina in the form of an ellipse is rotating in its own plane about one of its foci with angular velocity ω. This focus is set free and the other focus at the same instant is fixed. Show that the ellipse now rotates about it with angular velocity $\frac{\omega}{3}\left(\frac{2-5e^2}{2+3e^2}\right)$
69. An elliptic area of eccentricity e is rotating with angular velocity ω in about one latus rectum . suddenly this latus rectum is set free and the other at the same instant is fixed. Show that the ellipse now rotates about it with angular velocity $\omega\left(\frac{1-4e^2}{1+4e^2}\right)$
70. Three equal uniform rods placed in a straight line in a straight line are jointed at junctions and move with velocity v perpendicular to their lengths. If the middle point of the middle rod be suddenly fixed, show that the ends of other two rods will meet in time $\frac{4\pi a}{9v}$

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